Christopher Jekeli

Inertial Navigation Systems with Geodetic Applications

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Author
Dr. Christopher Jekeli
Department of Civil & Environmental
Engineering & Geodetic Science
The Ohio State University
470 Hitchcock Hall, 2070 Neil Avenue
Columbus, Ohio 43210-1275
USA

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To my Mother and Father

Preface

It is amazing that gyros perform as well as they do.

Wallace E. Vander Velde, M.I.T., 1983.

The quote above is a beautiful expression and tribute to the engineering triumph in the latter half of the twentieth century of the mechanical gyroscope that is an integral part of the traditional inertial navigation system (INS). In today's technological age of lasers and digital electronics, having also benefited inertial sensors, high performance is almost expected; yet the most accurate navigation and guidance systems still rely on the mechanical predecessors of the modern optical gyro. On the other hand, robustness, reliability, efficiency, and, above all, cost-effectiveness belong to the new generation of sensors and open the opportunity for increased utility and wider application in commercial, industrial, and scientific endeavors, Concurrent with the technological innovations came new analytical tools, specifically the Kalman filter, that is tailor-made for the analysis, calibration, and integration of the inertial navigation system. In the last two decades, the Global Positioning System (GPS) has come to dominate the wide range of positioning and navigation applications, but the new inertial sensor technology has enabled a continuing and growing utilization of these marvelous instruments and has motivated a revival from a number of different perspectives, the geodetic one in the present case, especially in regard to their integration with GPS.

Geodesy, the science of the measurement and determination of the Earth's surface, now routinely relies on the exquisite accuracy extractable from GPS, but equally recognizes the advantages and opportunities afforded by including INS in several applications. Although the present book is ultimately dedicated to this goal, it should prove to be of equal value to non-geodesists who desire a thorough understanding of the mathematics behind the INS and its general use for precision navigation and positioning. In particular, an attempt is made with this book to join the principles of inertial technology and estimation theory that applies not only to an understanding of the dynamics of the sensors and their errors, but to the integration of INS with other systems such as GPS. The text is written by a geodesist who loves the application of mathematics and believes that an appreciation of these topics comes best with illustrative formulas that are derived from first principles. As such, considerable effort is devoted to establishing preliminary concepts in coordinate systems, linear differential equations, and stochastic processes, which can also be found readily in other eminent texts, but whose inclusion makes the present text essentially self-contained. With the mathematical details, and occasional numerical considerations, it is also hoped that the reader will obtain an appreciation for Vander Velde's statement, above, that may well be extended to the navigation system and further to its entire technological development-the achievements in inertial technology are truly amazing.

The text assumes that the reader is fluent in the differential and integral calculus, as well as the basic vector and matrix algebras. Undergraduate courses in calculus, including the traditional advanced calculus, and linear algebra are, therefore, prerequisites to this text. Further minimal background in complex variables, differential equations, and numerical and statistical analyses, though not essential, would provide the reader with additional mathematical maturity to fully benefit from the material. The book may be used as a text for a college semester course at the senior or graduate level. Although mathematical derivations are given in detail, the old adage that mathematics is not a spectator sport should be followed by serious readers.

This text would not have materialized without the significant inspiration derived from my colleagues while working at the Air Force and at the Ohio State University. I would like to thank, in particular (affiliations are not necessarily current): Warren Heller, Jim White, Jacob Goldstein, Robert Shipp (TASC); Triveni Upadhyay, Jorge Galdos (Mayflower Communications, Inc.); David Gleason, Gerald Shaw (Air Force Geophysics Laboratory); Jim Huddle (Litton Guidance and Control, Inc.); Alyson Brown (NAVSYS Corp.); Klaus-Peter Schwarz (University of Calgary); Clyde Goad, Burkhard Schaffrin, Dorota Grejner-Brzezinska, C.K. Shum, Ren Da, Jin Wang, Jay Kwon (Ohio State University). In addition, several outstanding lecture notes by W. Vander Velde and A. Willsky (both at M.I.T.) have motivated key aspects of the mathematical understanding embodied in this text.

Columbus, Ohio; July 2000 C. Jekeli

Contents

About the Author

Christopher Jekeli received his Ph.D. degree in Geodetic Science at the Ohio State University in 1981, after which he was employed as research scientist at the Air Force Geophysics Laboratory, Bedford, Massachusetts, until 1993. There he led analysis teams associated with the airborne gravity gradiometer projects sponsored by the Defense Mapping Agency (now National Imagery and Mapping Agency) and non-Newtonian gravity experiments. In 1993 he became associate professor at Ohio State University (professor since 1998) in the Department of Geodetic Science and Surveying, now the Department of Civil and Environmental Engineering and Geodetic Science. His teaching and research interests and specializations include global and local gravity modeling and measurement, geoid determination, gravimetric spectral analysis, satellite geodesy, inertial navigation systems, GPS, and geodetic reference systems.

He is a Fellow of the International Association of Geodesy (IAG) and serves on its executive committee and as president of study groups in airborne gravimetry and the theory of height systems. Christopher Jekeli was a member of the National Research Council Committee on Geodesy and holds membership with the American Geophysical Union and the Institute of Navigation. He was associate editor of the *Journal of Geodesy* (formerly *Bulletin Géodésique* and *Manuscripta Geodaetica*) and publishes regularly in the fields of gravimetric geodesy and theory.

1 Coordinate Frames and Transformations

1.1 Introduction

When describing locations of points on or near the Earth's surface, we most naturally turn to a system of coordinates. Although one could imagine devising a relational or synthetic data base to describe the whereabouts of objects and places, it is of necessity that we assign an algebraic system of coordinates if we wish to go beyond mere location information and obtain measures of distance, area, volume, and direction. And, likewise with navigation, we need to define a coordinate system in which we can measure our progress and easily determine our course and destination. There are several coordinate systems from which to choose. Each has its own unique utility depending on the particular application in a particular discipline. In geodesy we deal with determining positions, or the mathematics of map projections, or the navigation of a vehicle, or its guidance along a predefined path. Specific coordinate systems must be defined in each case.

We will be concerned primarily with the *Cartesian*, or *rectangular*, coordinates, whose axes are mutually orthogonal by definition, but this triad of axes, as shown in Figure 1.1, may assume a variety of orientations in space. The axes of any coordinate system are designated here generally in numerical order as the 1-axis, the 2-axis, and the 3-axis. Each system of axes is defined to be right-handed in the sense that a 90°-counterclockwise (positive) rotation about the 1-axis, as viewed along the 1-axis toward the origin, rotates the 2-axis into the 3-axis. Also, a 90° rotation about the 2-axis rotates the 3-axis into the 1-axis; and, a 90° rotation about the 3-axis rotates the 1-axis into the 2-axis.

We will denote the set of Cartesian coordinates by a lower-case subscripted letter, such as x_j, $j = 1, 2, 3$. The corresponding bold letter, $\boldsymbol{x}$, will denote a vector with x_j as components (see Figure 1.1). Also shown in Figure 1.1 is a set of special vectors, called *unit vectors*, denoted by $\boldsymbol{e}_j$, $j = 1, 2, 3$. Each unit vector has only one nonzero component, namely the j-th component equals 1; that is, $\boldsymbol{e}_j$ is directed along its respective axis and has unit length.

The vector representation of $\boldsymbol{x}$ as an ordered triplet of coordinates is

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{1.1}$$

Or, using the unit vectors, we may also write:

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3, \tag{1.2}$$

where it is clear that the coordinates, x_j, are the orthogonal projections of $\boldsymbol{x}$ onto the respective axes. It is assumed that the reader is familiar with vector (and matrix) algebra, including the usual operations of addition, subtraction, and multiplication

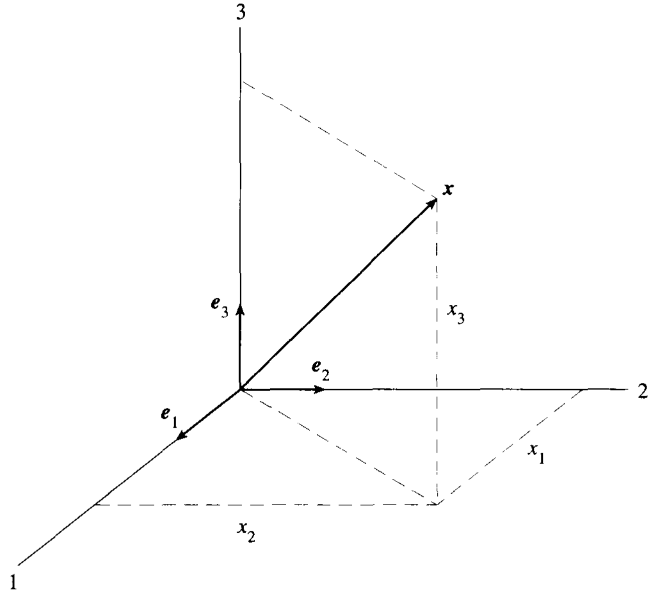

Figure 1.1: Cartesian coordinates of vector $\boldsymbol{x}$, and unit vectors $\boldsymbol{e}_j$.

(scalar or dot product, vector or cross product, as well as matrix multiplication and inversion). For a review of these algebras, see Lang (1971).

Before proceeding, a few additional introductory remarks are in order. The geodetic/astronomic conventional usage of the terms coordinate *system* and coordinate *frame* will be followed (Moritz and Mueller, 1987). The *system* includes the conventions and physical theories, or their approximations and models, that are used to define the triad of coordinate axes; while a *frame* denotes the accessible realization of the system through a set of points whose coordinates are monumented or otherwise observable. The general principles and methodologies of determining a coordinate (or, reference) system are beyond the scope of this text, and it is enough to know for present purposes that each set of coordinate axes is well-defined and accessible, and represents a coordinate frame. In the navigation literature reference is often made to frames instead of systems, a convention also adhered to in this text, where we will understand that for each frame there is a system in the background.

The frames that we will deal with are either global or local in extent, where the "extent" is largely defined by the application and the accuracies desired. The global Cartesian coordinates are tied either to the rotating Earth or to the celestial sphere ("fixed" stars); the local Cartesian coordinates are defined by local directions, such as north, east, and down. In addition, we consider curvilinear coordinates as alternative descriptors of points in space—we do not use these to characterize the intrinsic properties of surfaces (or space) which is perhaps their more pertinent mathematical (differential geometric) application; rather, their use is motivated by the fact that our motions and positions are generally close to being on a spherical or ellipsoidal surface. These coordinates are still three-dimensional and serve mostly to facilitate computations and derivations, besides having their own historical significance. The curvilinear coordinates are either the spherical polar coordinates, or more appropriately, the geodetic coordinates latitude, longitude, and height (these will be defined precisely in Section 1.2).

When considering inertial navigation systems, various additional coordinate frames enter; for example, those of the navigation instruments, of the platform on which they are mounted, and of the vehicle carrying the platform. Each of these is almost always a Cartesian frame. One of the main problems, as might already be apparent, is to relate each frame to all other relevant frames so that measurements in one frame can be transferred, or formulated, in the frame most appropriate to the end use of the information they provide. This entails the topic of *transformations* (Section 1.3). First, we define each coordinate frame more explicitly.

1.2 Coordinate Frames

Our application of coordinate frames centers on *geodesy*, the science of the measurement, determination, and mapping of coordinates of points on or near the Earth's surface. We extend this abbreviated, formal definition, paraphrased from a definition due to F.R. Helmert (1880), by including the kinematic and dynamic positioning of points along a trajectory, that is, *navigation*. By navigation we usually mean positioning in real time, or almost instantaneously; but, navigation systems clearly can be used for positioning in a post-mission application. In either case, by concerning ourselves with motion, we immediately invoke certain basic physical laws; and, as we will see, *gravitational* acceleration plays a significant role.

1.2.1 Inertial Frame

We start with the most fundamental coordinate system in geodesy, appropriately, the *inertial system*, defined classically as that system in which Isaac Newton's laws of motion hold. The Newtonian definition of the inertial system implies a Euclidean (Galilean) system, defined as a system with coordinates satisfying Euclidean geometry. In such a system, a body at rest (or, in uniform rectilinear motion) will remain at rest (respectively, in uniform rectilinear motion) in the absence of applied forces. This is Newton's *First Law of Motion*.

In addition, the dynamics of the motion in this system can be formulated on the basis of Newton's Second and Third Laws of Motion. Specifically, Newton's *Second Law of Motion* (Goldstein, 1950) states that the time-rate of change of the linear momentum of a particle equals the sum of applied forces, $\boldsymbol{F}$:

$$\boldsymbol{F} = \frac{d}{dt}(m_i \dot{\boldsymbol{x}}), \tag{1.3}$$

where $m_i\dot{\boldsymbol{x}}$ is the particle's linear momentum, being the product of $\dot{\boldsymbol{x}}$, its velocity, and m_i, its inertial mass. (Dots above a variable denote differentiation with respect to time—once with one dot, twice with two dots, etc.) If no forces act on the particle, its linear momentum is a constant (*Law of Conservation of Linear Momentum*).

Assuming the mass is a constant, we also have a slightly more familiar, though less correct form of Newton's Second Law,

$$m_i\ddot{\boldsymbol{x}} = \boldsymbol{F}, \tag{1.4}$$

showing that the acceleration, $\ddot{\boldsymbol{x}}$, of a particle in an inertial system is directly proportional to $\boldsymbol{F}$. Besides providing the foundation for the inertial coordinate system, these classical laws of Newton's form the basis for describing the dynamics of the inertial measurement units (Chapter 3).

In our world, a global inertial system is at best an abstraction, since any frame in the vicinity of the solar system is permeated by a gravitational field that possesses spatially varying gradients. For example, in the frame attached to the center of mass of the solar system and assumed to be non-rotating, a body initially at rest or in uniform rectilinear motion will accelerate under the gravitational influence of the sun and planets (thus violating Newton's First Law); and therefore, this frame is not inertial. The gravitational acceleration, like the centrifugal or coriolis acceleration, is not due to an externally applied physical action as meant by $\boldsymbol{F}$ in (1.3), but rather it is a consequence of a field, in this case, the gravitational field, and belongs to the class of *kinematic forces* (Martin, 1988) that cause accelerations that are independent of the mass being accelerated.

In order to proceed, then, with the classical approach, it is necessary to modify Newton's Second Law (just as it is sometimes modified in a rotating frame to account for centrifugal and coriolis accelerations). Equation (1.4) is changed to account for the acceleration due to an ambient gravitational field:

$$m_i\ddot{\boldsymbol{x}} = \boldsymbol{F} + m_g\boldsymbol{g}. \tag{1.5}$$

The gravitational acceleration vector, $\boldsymbol{g}$, can be thought of as the "proportionality factor" between the gravitational mass, m_g, and the resulting gravitational force, $\boldsymbol{F}_g$, as formulated by Newton's *Law of Gravitation* (inverse square law):

$$\boldsymbol{F}_g = k\frac{m_g M}{\ell^2}\boldsymbol{e}_\ell = \boldsymbol{g}m_g, \tag{1.6}$$

where the gravitational field is generated by the particle mass, M, a distance ℓ from m_g, and where $\boldsymbol{e}_\ell$ is the unit vector along the line joining the two particles and k is a constant (Newton's gravitational constant) that accommodates our particular choice of units.

With the (weak) Principle of Equivalence one may speak of just one kind of mass, $m_i = m_g = m$, and we have

$$\ddot{\boldsymbol{x}} = \boldsymbol{a} + \boldsymbol{g}, \tag{1.7}$$

where $\boldsymbol{a} = \boldsymbol{F}/m$ is the acceleration due to an applied force, or a force of physical action; $\boldsymbol{a}$ is also known as the *specific force* (force per unit mass). Examples of specific forces include atmospheric drag, the lift provided by an aircraft wing, and the "reaction force" the Earth's surface exerts on us to keep us from falling toward its center.

A note on terminology is in order here. Appearing in the geodetic literature (e.g., Moritz and Mueller, 1987) is the designation *quasi-inertial* frame, referred to as the Earth-centered frame accelerating, but not rotating, around the sun, being "practically inertial" because, relativistically speaking, the gravitational field of the solar system is relatively weak and the curvature of space-time is very small so that classical Newtonian dynamics hold. While this nomenclature may be adequate for the frame itself, it is unsatisfactory when considering objects moving in this frame. For clearly, if inertial frames are defined by Newton's First Law of Motion, as was done at the beginning of this section, and since gravitational attraction is *not* an applied force—it is a field, or a part of the space we live in, then even in an approximate sense, we generally do not encounter inertial frames, as illustrated above. If we wish to continue with classical methods (as we do here), we must always modify Newton's laws to account for gravitation. Fortunately, because of the Principle of Equivalence, this is easy to do. But, strictly speaking these frames are not inertial, nor quasi-inertial. Perhaps a better nomenclature is *pseudo-inertial* frame (much like the pseudorange in the glossary of the Global Positioning System (GPS, see Chapter 9) that means a measured range containing the effect of a clock bias).

Recognizing and understanding its origins, we will retain the name inertial frame (or, *i-frame*) for the frame that is attached to the Earth's center, is in *free-fall*, and is not rotating. The frame is freely falling in the gravitational fields of the sun, moon, and planets. The $\boldsymbol{g}$ in (1.7) includes the Earth's gravitational acceleration, as well as the differences in solar, lunar, and planetary gravitational accelerations (tidal accelerations) relative to the Earth's center. The frame's orientation is fixed to the celestial sphere as realized by the observed directions of *quasars*, extremely distant celestial objects that have not shown any evidence of changing their relative orientation. The coordinates of a point in the *i*-frame are components of the vector designated $\boldsymbol{x}^i$. The superscript denotes the frame in which the coordinates are expressed.

The International Earth Rotation Service (IERS) establishes the "inertial" system, or the International Celestial Reference System (ICRS), using an adopted set of directions defined by extragalactic radio sources (McCarthy, 1996; Feissel and Mignard, 1998), such that the 3-axis (the North Celestial Pole) and the 1-axis (on the celestial equator) are close to traditional definitions. The system is realized by a catalogue of 608 quasars whose directions are determined using the technique of Very Long Baseline Interferometry and by the coordinates of some 120,000 stars published in the Hipparcos Catalogue. The ITRS (Section 1.2.2) is connected to the ICRS using adopted theories and conventions for the nutation and precession of the Earth's spin axis and orbit in space, as well as the motion of the pole with respect to the Earth's surface. Formally, the origin of the ICRS is the center of mass of the solar system in order to be consistent with the definition of dynamic time which is the time argument in the theories of planetary motion. This difference of origins for the i-frame has no practical consequence in this presentation.

1.2.2 Earth-Centered-Earth-Fixed Frame

Next we consider a frame that is fixed to the Earth; its origin is also at the Earth's center of mass. Its coordinate axes are defined by convention such that the 3-axis is a mean, fixed, polar axis; and, on the corresponding equator a zero-longitude is defined that specifies the location of the 1-axis (and the mean Greenwich meridian). The mean polar axis intersects the Earth's surface at the *Conventional International Origin* (CIO) and is close to the spin axis that deviates variously from this fixed point due to *polar motion* (Moritz and Mueller, 1987).

The designation of a coordinate vector in the so-called *Earth-centered-Earth-fixed* (ECEF) frame, or *e-frame*, is $\boldsymbol{x}^e$. Historically, Earth-fixed coordinate systems (not necessarily geocentric) were realized through the definition of a *geodetic datum* (Torge, 1991), whereby the adopted coordinates of one or more points (in the sense of an adjusted minimal constraint) on the Earth's surface served the function of defining the origin. Today, the IERS establishes the so-called *International Terrestrial Reference System* (ITRS), which is realized through satellite laser ranging systems, as well as GPS and other satellite systems; therefore, it is geocentric. The International Terrestrial Reference Frame (ITRF) is based on a set of globally distributed observatories whose coordinates are corrected also for crustal motion due to plate tectonics (McCarthy, 1996).

1.2.3 Navigation Frame

The frame that is commonly used to describe the navigation of a vehicle is a *local* coordinate frame. To define this properly, we first consider global (or, historically, also regional) geodetic reference systems. Conventional geodetic reference systems consist of curvilinear coordinates (ϕ, λ) that define the direction of the normal (i.e., the perpendicular) to an adopted ellipsoid of revolution (see Figure 1.2). The parameters of this ellipsoid, defining its scale and shape, are chosen so that it approximates the zero-height level surface (the *geoid*) of the Earth.

The *geodetic latitude*, ϕ, of a point is the angle in the meridian plane of the normal through the point, with respect to the equator, positive northward or negative southward. The *geodetic longitude*, λ, is the angle in the equatorial plane from the Greenwich meridian (containing the 1^e-axis) to the meridian plane of the point. The *ellipsoid height*, h, is the distance from the ellipsoid along the normal to the point. The coordinates (ϕ, λ, h) constitute an orthogonal set of coordinates. They could be used in place of the Cartesian coordinates, $\boldsymbol{x}^e$, to describe positions in the e-frame, provided that a particular ellipsoid has been specified and that it is geocentric (i.e., its origin is the Earth's center of mass). The transformation between (ϕ, λ, h) and $\boldsymbol{x}^e$ is given in Section 1.5.

The local system of coordinates may now be defined as a set of Cartesian coordinate axes, where the third axis is aligned with the ellipsoidal normal at a point, in the "down" direction, the first axis points due north (parallel to the tangent to the

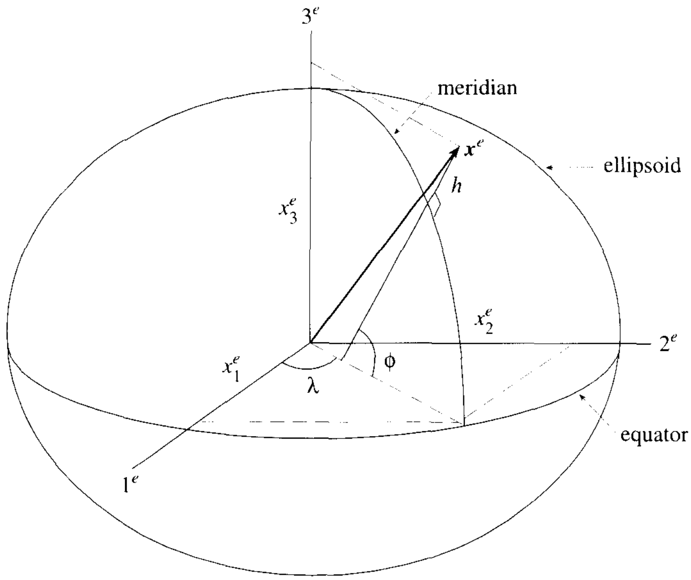

Figure 1.2: Earth-fixed-Earth-centered coordinates and geodetic coordinates with respect to an Earth Ellipsoid.

meridian), and the second axis points east (Figure 1.3). An alternative to north-east-down (NED) is south-east-up, where the positive 3-axis points up, and the frame is again right-handed. In contrast, geodesists often use local north-east-up frames for astronomic-geodetic observations; these are left-handed frames and will not be considered. The north-east-down frame, adopted here and conventionally implemented in the field of inertial navigation, is known as the *navigation frame*, or the *n-frame*. The origin of the n-frame is local, either on the ellipsoid, or at the location of the navigation system (as shown in Figure 1.3).

Note that the 3-axis of the n-frame does not pass through the Earth's center of mass. This adds a complication to the transformation of coordinates of points between the n-frame and the e-frame. But more important, the n-frame is not used generally to coordinatize a vehicle's *position*. Instead, one should visualize the n-frame such that the 3-axis always moves with the vehicle carrying the navigation system. The purpose of the n-frame primarily is to provide local directions, north, east, down, along which *velocities* may be indicated. This is particularly useful in those navigation systems that are mechanized such that the sensors are always aligned with the local horizon and vertical. As we will see, the same utility of the n-frame is offered through computational means to systems that are arbitrarily mechanized. In addition, the n-frame also serves as a common reference frame to which platform and sensor frames may be related.

With this special consideration of the n-frame, its relative *orientations* with respect to the e-frame and the i-frame are of primary importance. The vector $\boldsymbol{x}^n$ is not used to coordinatize the vehicle's position except in a formal sense because by definition only its third component might be non-zero.

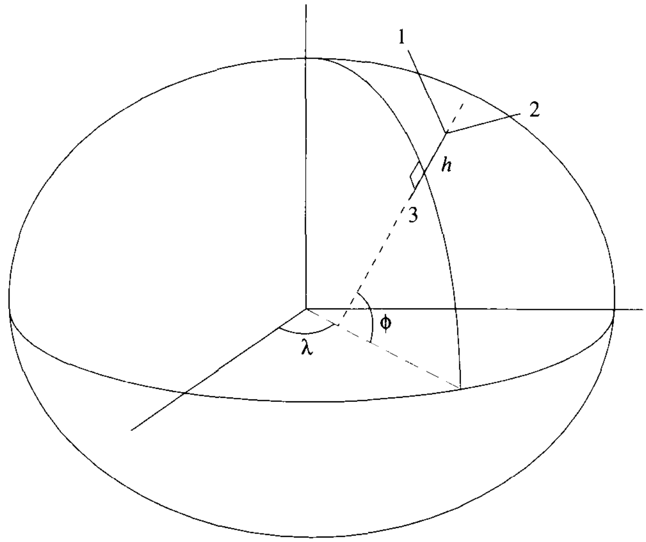

Figure 1.3: Local north-east-down coordinate frame.

In addition to these basic frames, one can associate, as already mentioned, Cartesian (orthogonal) coordinate frames with each of the various components of the measurement system and the vehicle carrying it (AFSC, 1992). The *body frame*, or *b*-frame, generally refers to the vehicle to be navigated. The axes conventionally are defined along the forward, right, and through-the-floor directions (see Figure 1.4, also Figure 1.8). The *sensor frame* is a single representative, analytic frame for the navigation system. It is used to model and identify instrument errors in a unified frame for analytical purposes like filtering. In the case of a strapdown system (Section 4.2.4) the sensor frame may be identified with the body frame, and in the case of a local-level gimbaled system (Section 4.2.2) the sensor frame usually corresponds to the navigation frame.

The instruments of an inertial navigation system (INS) are accelerometers and gyroscopes; the accelerometers measure accelerations that are integrated to yield positions and the gyroscopes provide information on the orientation of the accelerometers (see Chapter 3). Each set of instruments, accelerometers and gyroscopes, has its own coordinate system. The *accelerometer frame* is taken to be orthogonal, but with the realization that the sensitive axis of only one accelerometer will be aligned with a frame axis (1-axis), while the others may be misaligned with respect to the corresponding frame axes. This non-orthogonality of the instruments is determined with special calibration procedures. The origin of the accelerometer frame is the point of specific force computation for the accelerometers. The *gyro frame* similarly is orthogonal with only one of the input axes aligned along a frame axis (1-axis). The origin of the gyro frame is the same as the accelerometer frame. For purposes of understanding the operation of each instrument, we introduce